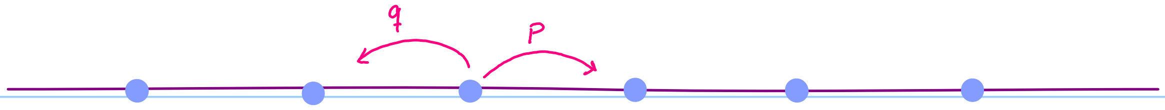
Quantum Algorithms; part III

Note Title 10/17/2010

- ## Classical Random Walk.



$$P_N(x) = p\,P_{N-1}(x-1) + q\,P_{N-1}(x+1)$$

$$Z_N(s) := \sum_{x=-\infty}^{\infty} e^{sx} P_N(x)$$

$$Z_N(0) = 1\ , \qquad \left.\frac{dZ_N}{ds}\right|_{s=0} = \langle X \rangle \qquad \left.\frac{d^2 Z_N}{ds^2}\right|_{s=0} = \langle X^2 \rangle$$

$$Z_N(s) = (p\,e^{s} + q\,e^{-s})\, Z_{N-1}(s)$$

$$Z\;(s) = (pe^{s} + qe^{-s})^{N}\, Z_0(s)$$

if $P_0(x) = \delta_{0,x} \longrightarrow Z_0(s) = 1$

$$Z(s) = (pe^{s} + qe^{-s})^{N}$$

$$\langle x\rangle = (p-q)N \qquad \Delta X := \sqrt{\langle X^2\rangle - \langle X\rangle^2} = \sqrt{Npq}$$

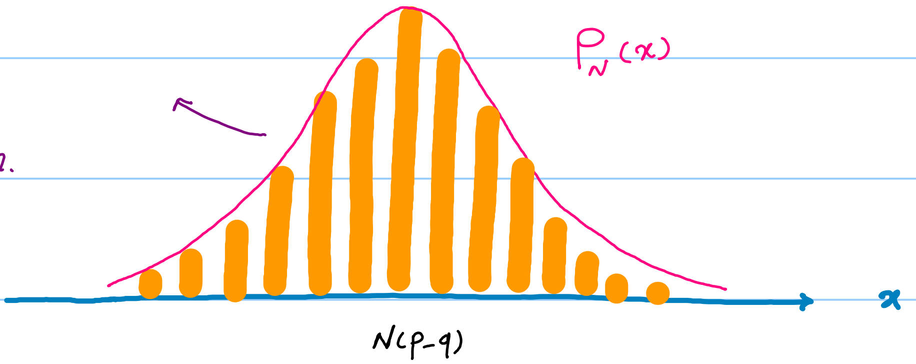
$$\begin{cases} n_+ - n_- = x \\ n_+ + n_- = N \end{cases} \longrightarrow \quad n_+ = \frac{N+x}{2} \qquad n_- = \frac{N-x}{2}$$

$$P_N(x) = \binom{N}{\frac{N+x}{2}} p^{\frac{N+x}{2}} q^{\frac{N-x}{2}}$$

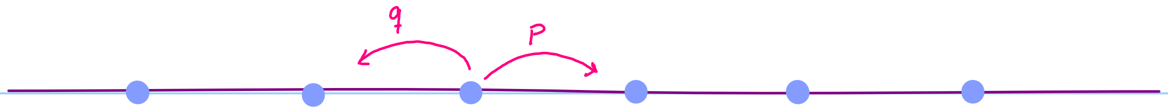
Binomial

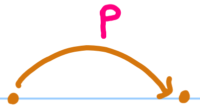
or

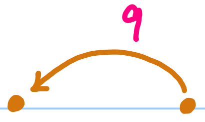
Gaussian.



- Quantum Random Walk



quantum Coin $|0\rangle$ 

$|1\rangle$ 

$$\mathcal{S} = \sum_n |n+1\rangle\langle n| \qquad \mathcal{S}^{-1} = \sum_n |n-1\rangle\langle n|.$$

$$\mathcal{U}_0 = \mathcal{S} \otimes |0\rangle\langle 0| + \mathcal{S}^{-1} \otimes |1\rangle\langle 1|$$

$$\mathcal{H}_c = \mathcal{H}_{coin} = \{ |\uparrow\rangle, |\downarrow\rangle \}$$

$$\mathcal{H}_s = \mathcal{H}_{lattice} = \{ |n\rangle , \quad n \in \mathbb{Z} \}$$

$$= \{ \dots |-3\rangle, |-2\rangle, |-1\rangle, |0\rangle, |1\rangle, |2\rangle \dots \}$$

$$\mathcal{U}_0 : \mathcal{H}_S \otimes \mathcal{H}_C \longrightarrow \mathcal{H}_S \otimes \mathcal{H}_C$$

Initial State of the Coin : $|+\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$

$$|\psi(0)\rangle = |m,+\rangle = \frac{1}{\sqrt{2}}\left(|m,\uparrow\rangle + |m,\downarrow\rangle\right)$$

$$|\psi(1)\rangle = \frac{1}{\sqrt{2}}\left(|m+1,\uparrow\rangle + |m-1,\downarrow\rangle\right)$$

$$|\psi(2)\rangle = \frac{1}{\sqrt{2}}\left(|m+2,\uparrow\rangle + |m-2,\downarrow\rangle\right)$$

$$|\psi(3)\rangle = \frac{1}{\sqrt{2}}\left(|m+3,\uparrow\rangle + |m-3,\downarrow\rangle\right)$$

در این حالت، ولگشت بوجود نمی آید، بلکه دو حرکت یعنی به سمت چپ و راست با هم ترکیب می‌شوند.

برای عمل سکه انداختن از یک عملگر در فضای سکه استفاده می‌کنیم. این عملگر را هادامارد انتخاب می‌کنیم و قرار می‌دهیم:

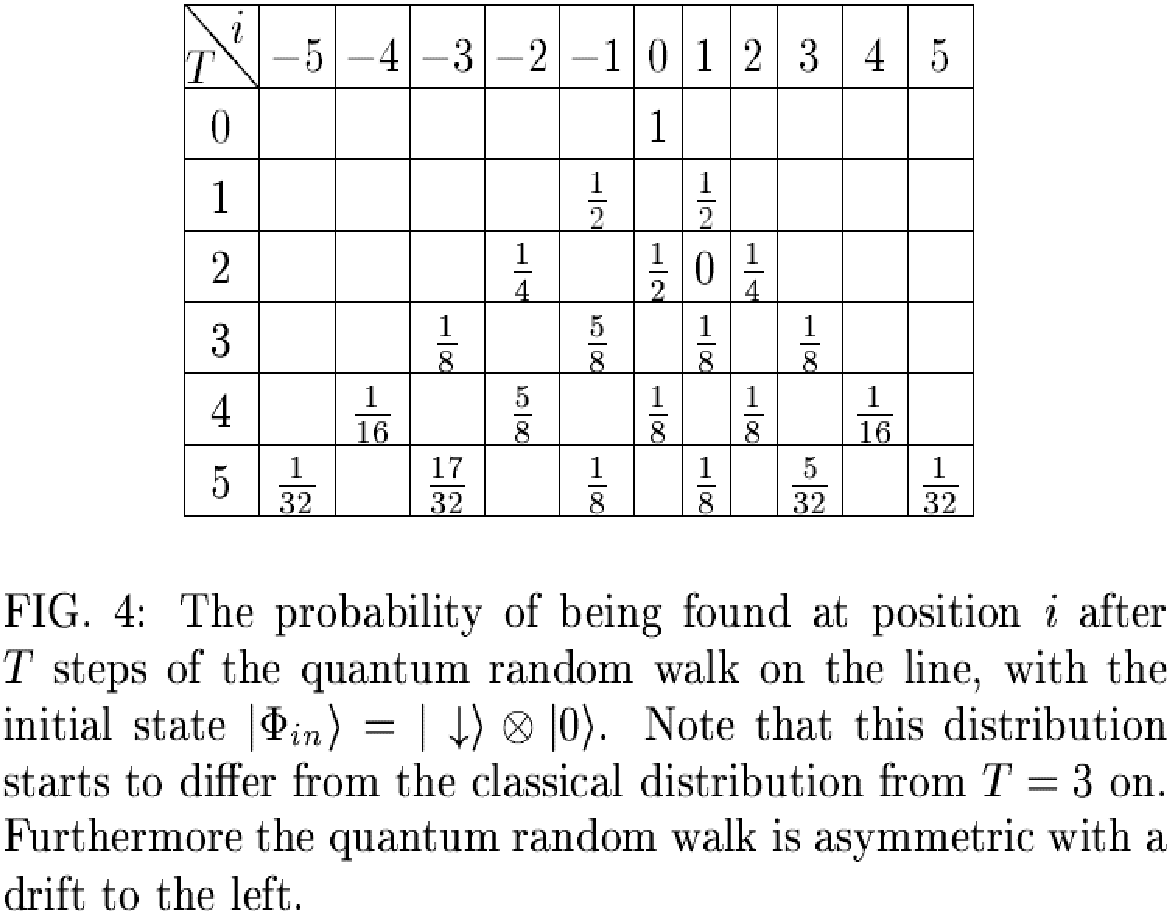
$$\mathcal{U} := \mathcal{U}_0 (I \otimes H)$$

$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $|\psi(0)\rangle = |0,\uparrow\rangle$

$$\boxed{|0,\uparrow\rangle} \xrightarrow{(1 \otimes H)} |0,\uparrow\rangle + |0,\downarrow\rangle \xrightarrow{U_0} \boxed{|1,\uparrow\rangle + |-1,\downarrow\rangle}$$

$$|1,\uparrow\rangle + |-1,\downarrow\rangle \xrightarrow{(1 \otimes H)} |1,\uparrow\rangle + |1,\downarrow\rangle + |-1,\uparrow\rangle - |-1,\downarrow\rangle \xrightarrow{U_0}$$

$$\boxed{|2,\uparrow\rangle + |0,\uparrow\rangle + |0,\uparrow\rangle - |-2,\downarrow\rangle} \longrightarrow \cdots$$



T \ i	−5	−4	−3	−2	−1	0	1	2	3	4	5
0						1					
1					$\frac{1}{2}$		$\frac{1}{2}$				
2				$\frac{1}{4}$		$\frac{1}{2}$	0	$\frac{1}{4}$			
3			$\frac{1}{8}$		$\frac{5}{8}$		$\frac{1}{8}$		$\frac{1}{8}$		
4		$\frac{1}{16}$		$\frac{5}{8}$		$\frac{1}{8}$		$\frac{1}{8}$		$\frac{1}{16}$	
5	$\frac{1}{32}$		$\frac{17}{32}$		$\frac{1}{8}$		$\frac{1}{8}$		$\frac{5}{32}$		$\frac{1}{32}$

FIG. 4: The probability of being found at position i after T steps of the quantum random walk on the line, with the initial state $|\Phi_{in}\rangle = |\downarrow\rangle \otimes |0\rangle$. Note that this distribution starts to differ from the classical distribution from $T = 3$ on. Furthermore the quantum random walk is asymmetric with a drift to the left.

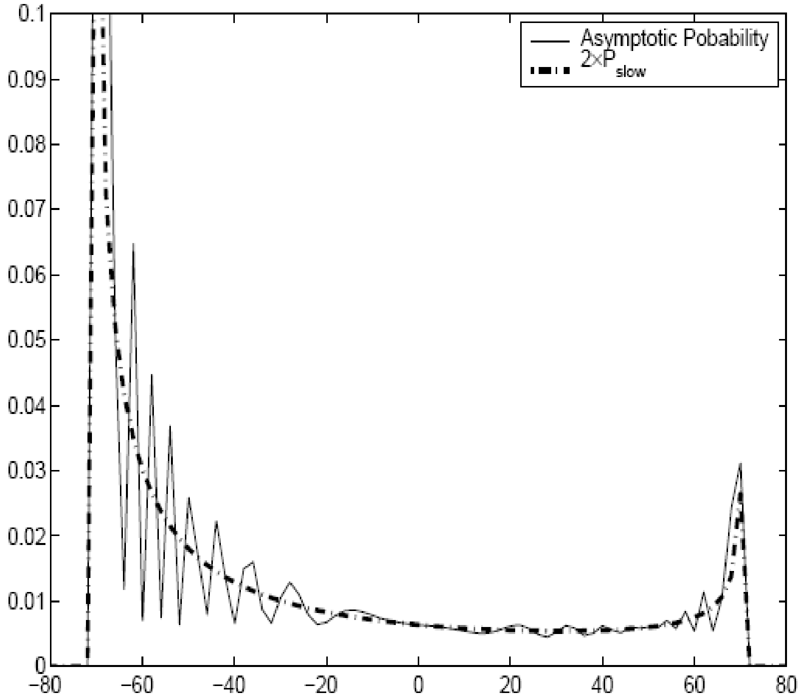


FIG. 5: The probability distribution of the quantum random walk with Hadamard coin starting in $|\downarrow\rangle \otimes |0\rangle$ after $T = 100$ steps. Only the probability at the even points is plotted, since the odd points have probability zero. The dotted line gives a long-wavelength approximation (labeled P_{slow} since it keeps only the slowly varying frequencies [15]), clearly showing the bi-modal character of the distribution.

برای یک ولگشت متقارن به دست آوردن یکی از دو راه زیر را می‌توانیم طی کنیم:

Symmetric Walk:

1) $|\uparrow\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + i|\downarrow\rangle\right)$

یا

2) $H \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$

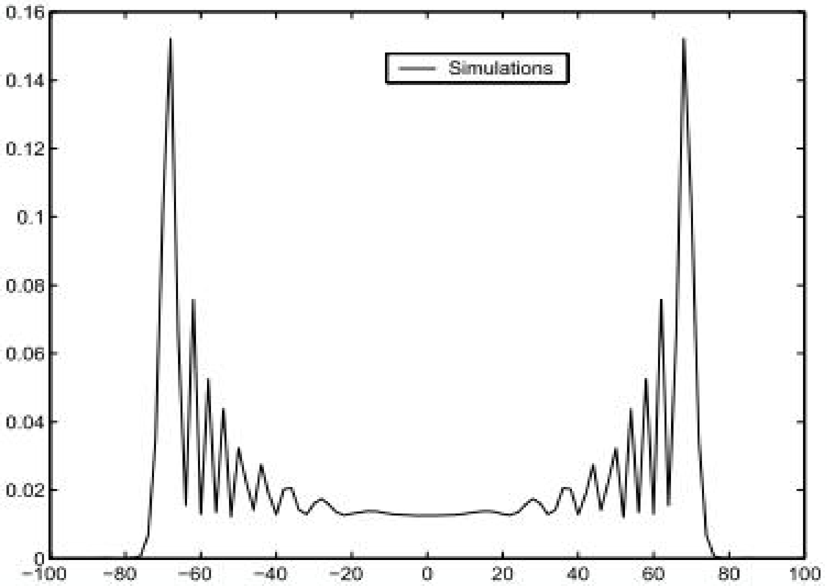
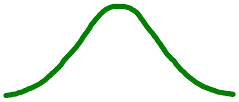
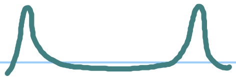


FIG. 6: The probability distribution obtained from a computer simulation of the Hadamard walk with a symmetric initial condition [15]. The number of steps in the walk was taken to be 100. Only the probability at the *even* points is plotted, since the odd points have probability zero.

	Variance	Distribution
classical	$\sigma \sim \sqrt{T}$	
quantum	$\sigma \sim T$	

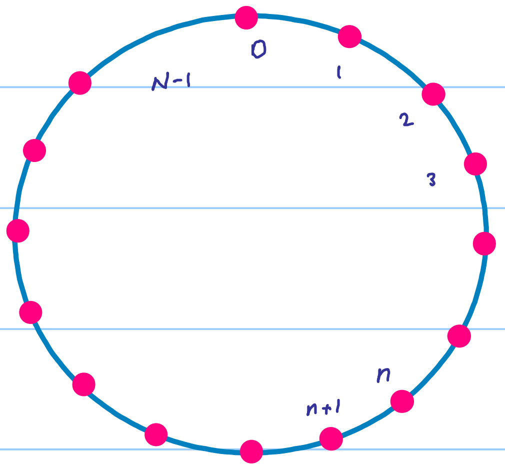
Analytical Solution

$$U = \left(S \otimes |0\rangle\langle 0| + S^{-1} \otimes |1\rangle\langle 1| \right) \left(I \otimes H \right)$$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} S & \\ & S^{-1} \end{bmatrix} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} S & S \\ S^{-1} & -S^{-1} \end{bmatrix}$$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} S & S \\ S^{-1} & -S^{-1} \end{pmatrix}$$

- ویژه‌مقادیر S :



$$S|x\rangle = |x+1\rangle \longrightarrow S^N |x\rangle = |x\rangle$$

$$S^N = I$$

$$\text{Eigenvalues of } S = \{ w^k \quad : k=0,1,\dots N-1 \}$$

$$w^N = 1$$

$$S\,|\phi_k\rangle = w^k\,|\phi_k\rangle \qquad |\phi_k\rangle = \sum_{x=0}^{N-1} C_x\,|x\rangle$$

$$\sum_{x=0}^{N-1} C_x\,|x+1\rangle = w^k \sum_{x=0}^{N-1} C_x\,|x\rangle \;\longrightarrow\; C_{x-1} = w^k\, C_x$$

$$\rightarrow\; C_x = w^{-kx}\, C_0 \;\longrightarrow\; \boxed{|\phi_k\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} w^{-kx}\,|x\rangle}$$

ویژه بردار S با ویژه مقدار w^k.

- بردارهای پایه جدید را به شکل زیر انتخاب می‌کنیم.

$$|\phi_k^+\rangle = \begin{bmatrix} |\phi_k\rangle \\ 0 \end{bmatrix} \qquad |\phi_k^-\rangle = \begin{bmatrix} 0 \\ |\phi_k\rangle \end{bmatrix}$$

$$\mathcal{U}\,|\phi_k^+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} S & S \\ S^{-1} & -S^{-1} \end{bmatrix} \begin{bmatrix} |\phi_k\rangle \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \omega^k\,|\phi_k\rangle \\ \omega^{-k}\,|\phi_k\rangle \end{bmatrix}$$

در این بردارهای پایه، اثر $\mathcal{U}$ خیلی ساده‌تر است.

$$\mathcal{U}\,|\phi_k^+\rangle = \frac{1}{\sqrt{2}}\left(w^k\,|\phi_k^+\rangle + w^{-k}\,|\phi_k^-\rangle\right)$$

$\rightarrow$

$$\mathcal{U}\,|\phi_k^-\rangle = \frac{1}{\sqrt{2}}\left(w^k\,|\phi_k^+\rangle - w^{-k}\,|\phi_k^-\rangle\right)$$

$$\mathcal{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} w^k & w^k \\ w^{-k} & -w^{-k} \end{bmatrix}$$ in the subspace of $|\phi_k^+\rangle, |\phi_k^-\rangle$.

فعلاً $\frac{1}{\sqrt{2}}$ را در نظر نمی‌گیریم.

$$(w^k - \lambda')(-w^{-k} - \lambda') - 1 = 0 \;\rightarrow\; \lambda'^2 + \lambda'(w^{-k} - w^k) - 2 = 0$$

$$\lambda'^2 - 2i\lambda' \sin\frac{2\pi}{N}k - 2 = 0 \;\rightarrow\; \lambda' = i\sin\frac{2\pi}{N}k \pm \sqrt{2 - \sin^2\frac{2\pi}{N}k}$$

$$\lambda_k^{\pm} = \frac{1}{\sqrt{2}}\left(i \sin\frac{2\pi}{N}k \pm \sqrt{1+\cos^2\frac{2\pi}{N}k}\right) \rightarrow |\lambda_k^{\pm}| = 1$$

$$\mathcal{U}\,|e_k^{\pm}\rangle = \lambda_k^{\pm}\,|e_k^{\pm}\rangle$$

$$|\psi(0)\rangle = \sum_{k=0}^{N-1} C_k^{+}(0)\,|e_k^{+}\rangle + C_k^{-}(0)\,|e_k^{-}\rangle$$

$$|\psi(t)\rangle = \sum_{k=0}^{N-1} C_k^{+}(0)\,(\lambda_k^{+})^{t}\,|e_k^{+}\rangle + C_k^{-}(0)(\lambda_k^{-})^{t}\,|e_k^{-}\rangle$$

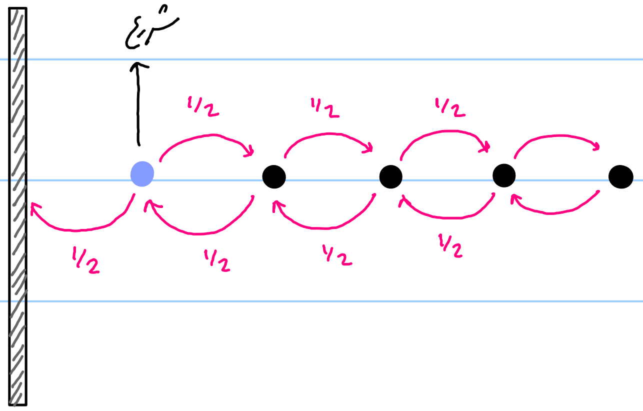
$$P_t(x,0) = |\langle x,0|\psi(t)\rangle|^2$$

$$P_t(x,1) = |\langle x,1|\psi(t)\rangle|^2$$

$$\boxed{P_t(x) = \mathrm{tr}\left\{\left(|x\rangle\langle x|\otimes I\right)|\psi(t)\rangle\langle\psi(t)|\right\}.}$$

- دیواره جذب کننده (یک تفاوت دیگر با ولگشت هادامارد).

- Classical Walk

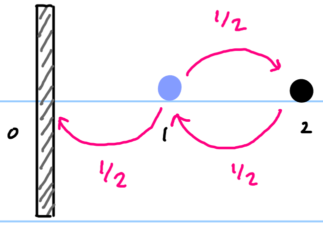


$P =$ احتمال جذب شدن به دیواره $= 1$

$P = P_{10} =$ احتمال اینکه ذره ای که در نقطه ۱ است سرانجام به نقطه 0 برسد

$P_{10} = P_{21} = P_{32} = \dots$ ← حالتی شبیه

رابطهٔ تکرار

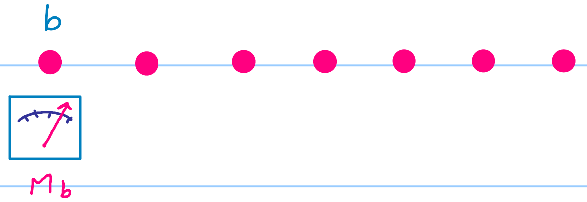


$$P_{10} = \frac{1}{2} + \frac{1}{2} P_{20}$$

$$P_{20} = P_{21} P_{10} = P_{10}^{2}$$

$$p = \frac{1}{2}(1+p^{2}) \rightarrow p = 1$$

- Quantum Walk



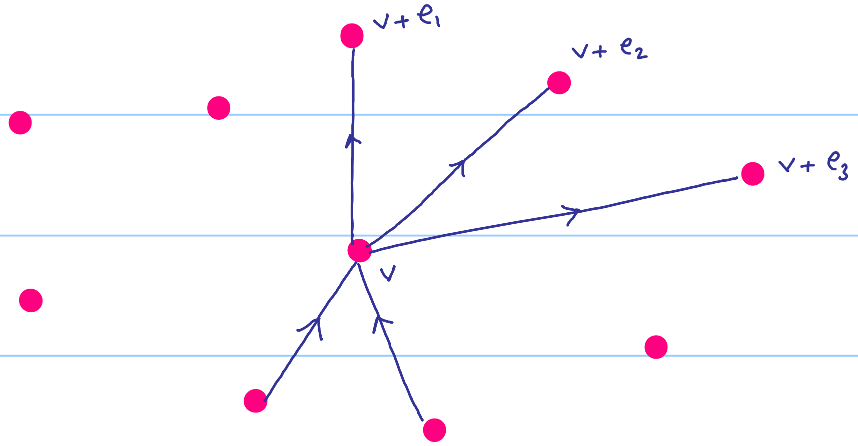
$$\mathcal{U} = (M_b \otimes I)\left(S \otimes |0\rangle\langle 0| + S^{-1} \otimes |1\rangle\langle 1|\right)(I \otimes H)$$

$$M_b|\psi\rangle = \begin{cases} |b\rangle & P_b = |\langle b|\psi\rangle|^{2} \\ \dfrac{|\psi\rangle - |b\rangle\langle b|\psi\rangle}{\sqrt{1-|\langle b|\psi\rangle|^{2}}} & 1-P_b \end{cases}$$

- Quantum Walk ⤵

$$P_{escape} = 1 - \frac{2}{\pi} \quad , \quad P_{abs} = \frac{2}{\pi}$$

- Quantum Walks on Arbitray Graphs.



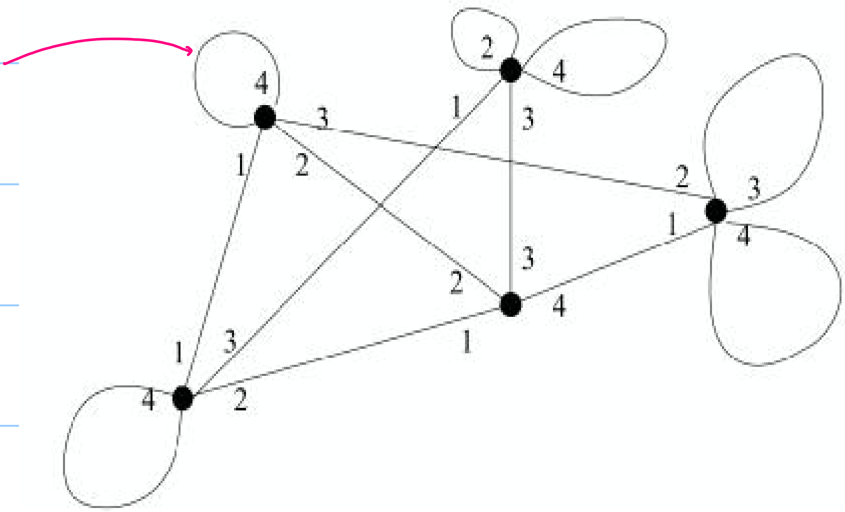
$$\mathcal{U}_v = |1\rangle\langle 1| \otimes |v+e_1\rangle\langle v| + |2\rangle\langle 2| \otimes |v+e_2\rangle\langle v| + |3\rangle\langle 3| \otimes |v+e_3\rangle\langle v| + \cdots$$

$$\mathcal{U} = \sum_{j,v} |j\rangle\langle j| \otimes |v+e_j\rangle\langle v|$$

با اضافه کردن حلقه ٤

می‌توانیم درجهٔ همه حلقه ٤ ها

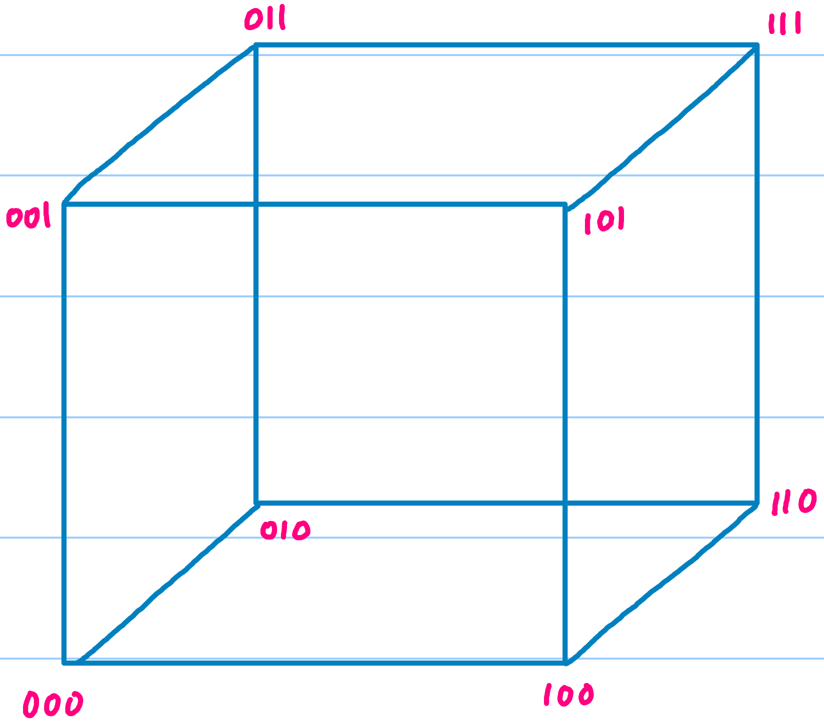
مساوی کنیم.



Balanced Coin:

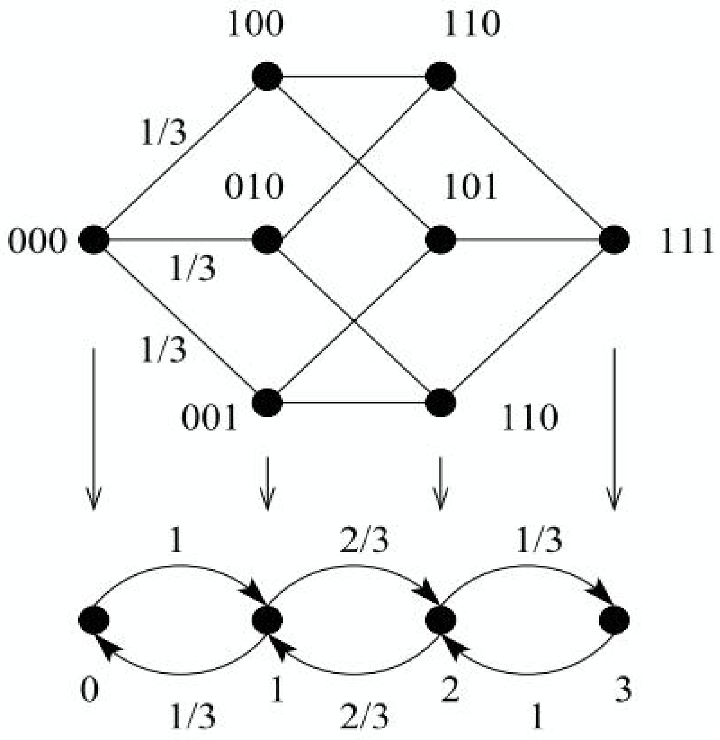
$$DFT = \frac{1}{\sqrt{d}}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{d-1} \\ & & \dots & & \\ 1 & \omega^{d-1} & \omega^{2(d-1)} & \dots & \omega^{(d-1)(d-1)} \end{pmatrix}.$$

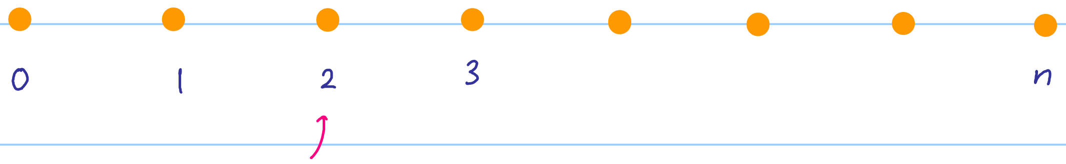
- An example of a biased coin.



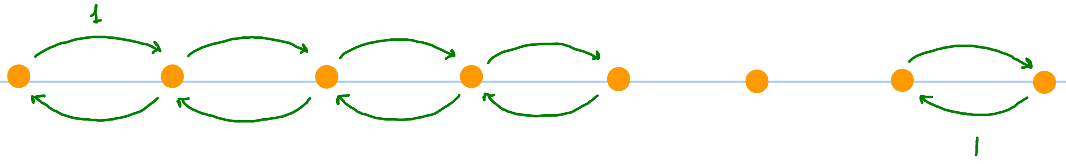
Hamming distance d_H.

$$n\text{-}\,\text{Hypercube} = \{0,1\}^n = \{ s_1 s_2 \ldots s_n \;:\; s_i = 0,1 \}$$



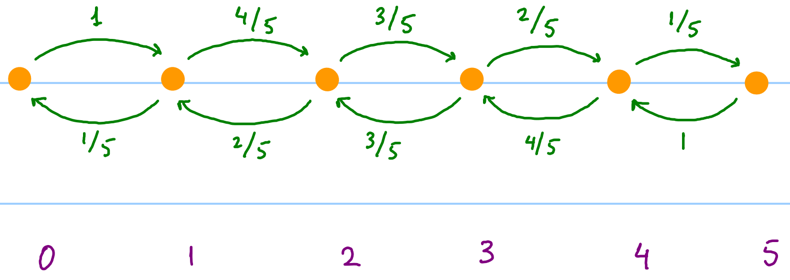


Hamming weight { 11000..0, 011000..0, 0101000..}.



$$w = \underbrace{1\;1\;0\;0\;1\;0\;0\;1\;0\;1\;1\;1\;0}_{n} \qquad d_w = 7$$

$$P_{d\to d+1} = \frac{n-d}{n} \qquad\qquad P_{d\to d-1} = \frac{d}{n}$$



- **Continuous Time Walk.** • ولگشت در زمان پیوسته
  - Master Equation in Continuous Time.

$$\mathbb{P}_n(c) = \sum_{c'} p(c|c')\,\mathbb{P}_{n-1}(c')$$

$$\sum_{c} p(c|c') = 1 \qquad\qquad |\mathbb{P}_n\rangle = \begin{bmatrix} P_n(c_1) \\ P_n(c_2) \\ \vdots \\ \end{bmatrix}$$

$$\Rightarrow \quad |\mathbb{P}_n\rangle = Q\,|\mathbb{P}_{n-1}\rangle$$

$$Q = \begin{bmatrix} p(c_1|c_1) & p(c_1|c_2) & & p(c_1|c_N) \\ p(c_2|c_1) & & & \\ & & & \\ p(c_N|c_1) & & & \end{bmatrix}$$

Now: $n\epsilon = t \qquad (n+1)\epsilon = t + \epsilon$ پیوسته کردن

$$\boxed{\mathbb{P}_n(c) = P(t, c)}$$

$$\begin{cases} p(c|c') = \epsilon\, w(c|c') & c' \neq c \\ p(c'|c') = 1 - \sum\limits_{c \neq c'} p(c|c') = 1 - \epsilon \sum\limits_{c \neq c'} w(c|c') \end{cases}$$

$$\mathbb{P}_{n+1}(c) = \sum_{c'} p(c|c')\, \mathbb{P}_n(c')$$

$$\mathbb{P}(t+\epsilon, c) = \sum_{c' \neq c} p(c|c')\, \mathbb{P}(t, c) + p(c|c)\, \mathbb{P}(t, c)$$

$$P(t,c) + \epsilon \frac{\partial P}{\partial t}(t,c) = \sum_{c' \neq c} \epsilon\, w(c|c')\, P(t,c') + \Big(1 - \epsilon \sum_{c' \neq c} w(c'|c)\Big) P(t,c)$$

$$\frac{\partial}{\partial t} P(t,c) = \sum_{c' \neq c} w(c|c')\, P(t,c') - \sum_{c' \neq c} w(c'|c)\, P(t,c)$$

$$\frac{\partial}{\partial t} P(t,c) = \sum_{c'} H(c,c')\, P(t,c')$$

$$\frac{\partial}{\partial t} |P(t)\rangle = H |P(t)\rangle$$

$>0.$

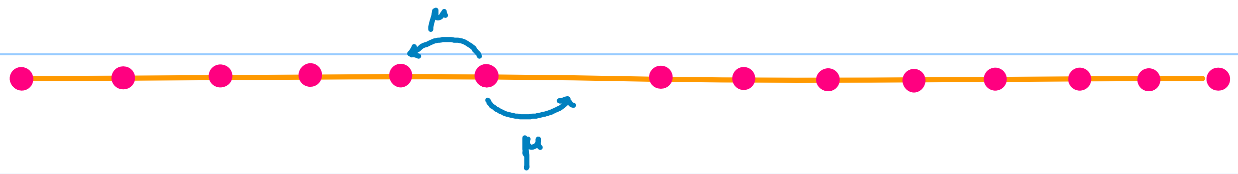
<0

$$H = \begin{bmatrix} w_{11} & w_{12} & w_{13} & & w_{1N} \\ w_{21} & w_{22} & w_{23} & & \\ w_{31} & w_{32} & w_{33} & & \\ & & & & \\ w_{N1} & w_{N2} & w_{N3} & & w_{NN} \end{bmatrix}$$

$$\sum_{i} H_{ij} = 0$$

$$\frac{\partial}{\partial t} |P(t)\rangle = H |P(t)\rangle$$

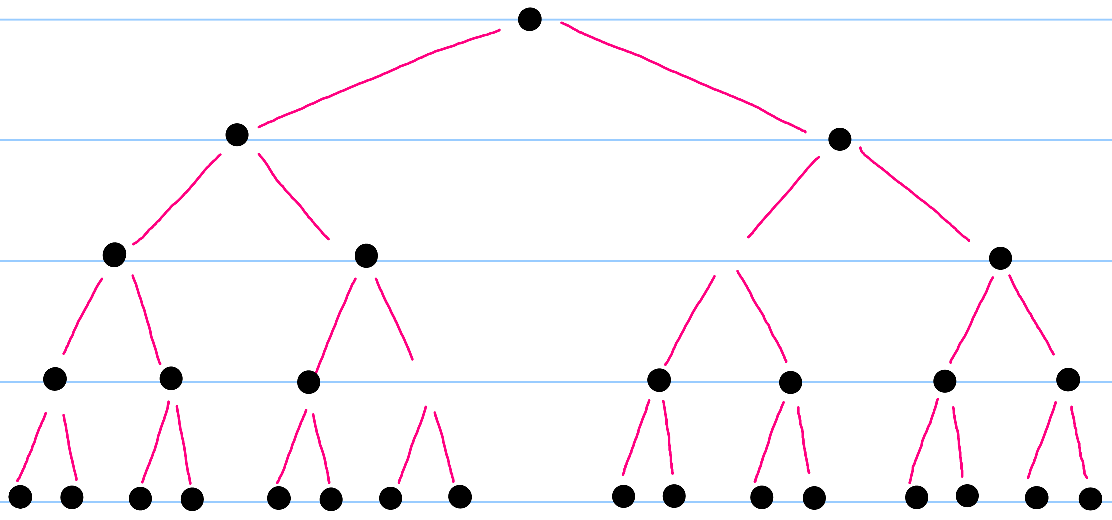
- The Matrix H for Random walk on a line.



$$H = \mu \begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & & & \\ & 1 & -2 & & \\ & & & & \\ & & & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}$$

- Continuous time

classical Random walk on a Graph.



$$H_{ij} = \begin{cases} \gamma & \text{if } i \,\&\, j \text{ are connected} \\ 0 & \text{if } i \,\&\, j \text{ are not connected.} \\ -3\gamma & \text{if } i = j \end{cases}$$

H is symmetric & Real.

Classical Random Walk: $$\frac{d}{dt}\,|p(t)\rangle = H\,|p(t)\rangle$$

$\uparrow$

احتمالات

Quantum Random Walk: $$\frac{d}{dt}\,|\psi(t)\rangle = -i\,H\,|\psi(t)\rangle$$

$\uparrow$

دامنه احتمالات .

- خاصیت‌های بیشتر در مورد ولگشت‌های رملایک و گرانتوی

Mixing Time: $P_{t+1} = Q P_t$

Stationary Distribution: $p^* = Q p^*$

$$M_\epsilon = \text{mixing time} = \min_T \left\{ T \mid \forall t > T;\ \| P_t - p^* \| < \epsilon \right\}.$$

$$\frac{\lambda_2}{(1-\lambda_2)\log \epsilon} \leq M_\epsilon \leq \frac{1}{1-\lambda_2}\left\{\max_i \log \frac{1}{p_i^*} + \log \frac{1}{\epsilon}\right\}$$

- For a Random Walk on the Circle

$$M_\epsilon = N^2 \log \frac{1}{\epsilon}$$

Hitting Time $\approx N^2$

- For Hypercube $\{0,1\}^d$

$$M_\epsilon = d \log d \log \frac{1}{\epsilon}$$

exponential → $T = 2^d$

$$\frac{\partial}{\partial t} |P(t)\rangle = H |P(t)\rangle$$ Classical.

$$\frac{\partial}{\partial t} |\psi(t)\rangle = -i H |\psi(t)\rangle$$ Quantum

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle$$ No stationary state

$$|C(t)\rangle = \frac{1}{t} \sum_{s=0}^{t} |P(s)\rangle \qquad |P(s)\rangle = \begin{bmatrix} P(s,1) \\ P(s,2) \\ \vdots \\ P(s,N) \end{bmatrix}$$

$$P(s,i) = |\langle i,\uparrow|\psi(s)\rangle|^2 + |\langle i\downarrow|\psi(s)\rangle|^2$$

$$C_i(t) = \frac{1}{t} \sum_{s=1}^{t} P(s,i)$$

$$= \frac{1}{t} \sum_{s=1}^{t} \sum_{\alpha=\uparrow,\downarrow} |\langle i,\alpha|\psi(s)\rangle|^2$$

$$|\psi(s)\rangle = \mathcal{U}^{s} |\psi(0)\rangle$$

$$\mathcal{U} | u_k\rangle = \lambda_k |\mathcal{U}_k\rangle \qquad k = 1, 2, \ldots\ldots 2N$$

$$|\psi(0)\rangle = \sum_{k=1}^{2N} a_k |\mathcal{U}_k\rangle \longrightarrow |\psi(s)\rangle = \sum_{k=1}^{2N} a_k \lambda_k^{s} |\mathcal{U}_k\rangle$$

$$\left|\langle i;\alpha|\psi(s)\rangle\right|^2 = \sum_{k,l=1}^{2N} a_k \bar{a}_l \left(\frac{\lambda_k}{\lambda_l}\right)^{s} \langle \mathcal{U}_l | i;\alpha\rangle\langle i;\alpha|\mathcal{U}_k\rangle$$

$$C_i(t) = \frac{1}{t} \sum_{s=1}^{t} \sum_{\alpha=\uparrow,\downarrow} \sum_{k=1}^{2N} a_k \bar{a}_l \left(\frac{\lambda_k}{\lambda_l}\right)^{s} \langle \mathcal{U}_l | i;\alpha\rangle\langle i;\alpha|\mathcal{U}_k\rangle$$

$$\mathcal{A} = \frac{1}{t} \sum_{s=1}^{t} \left(\frac{\lambda_k}{\lambda_l}\right)^{s} = \frac{1}{t} \frac{\left(\frac{\lambda_k}{\lambda_l}\right) - \left(\frac{\lambda_k}{\lambda_l}\right)^{t+1}}{1 - \frac{\lambda_k}{\lambda_l}}$$

$$\lim_{t\to\infty} \mathcal{A} = \begin{cases} 1 & \text{if } \lambda_k = \lambda_l \\ 0 & \text{if } \lambda_k \neq \lambda_l \end{cases}$$

$$C_i(t) = \frac{1}{t}\sum_{s=1}^{t}\sum_{\alpha=\uparrow,\downarrow}\sum_{k,l=1}^{2N} a_k \bar{a}_l \left(\frac{\lambda_k}{\lambda_l}\right)^s \langle u_l | i;\alpha\rangle\langle i;\alpha | u_k\rangle$$

$$C_i^* := \lim_{t\to\infty} C_i(t) = \sum_{\alpha=\uparrow,\downarrow}\sum_{\substack{k,l\\ \lambda_k=\lambda_l}} a_k \bar{a}_l \langle u_l | i;\alpha\rangle\langle i;\alpha | u_k\rangle$$

- Nondegenerate eigenvalues

$$C_i^* = \sum_{\alpha=\uparrow,\downarrow}\sum_{k=1}^{2N} |a_k|^2 \langle u_k | i;\alpha\rangle\langle i;\alpha | u_k\rangle$$

- ①
  ولگشت کلاسیک . حالت پایا به شرایط اولیه بستگی ندارد.
  ولگشت کوانتومی : حالت پایا به شرایط اولیه بستگی دارد.

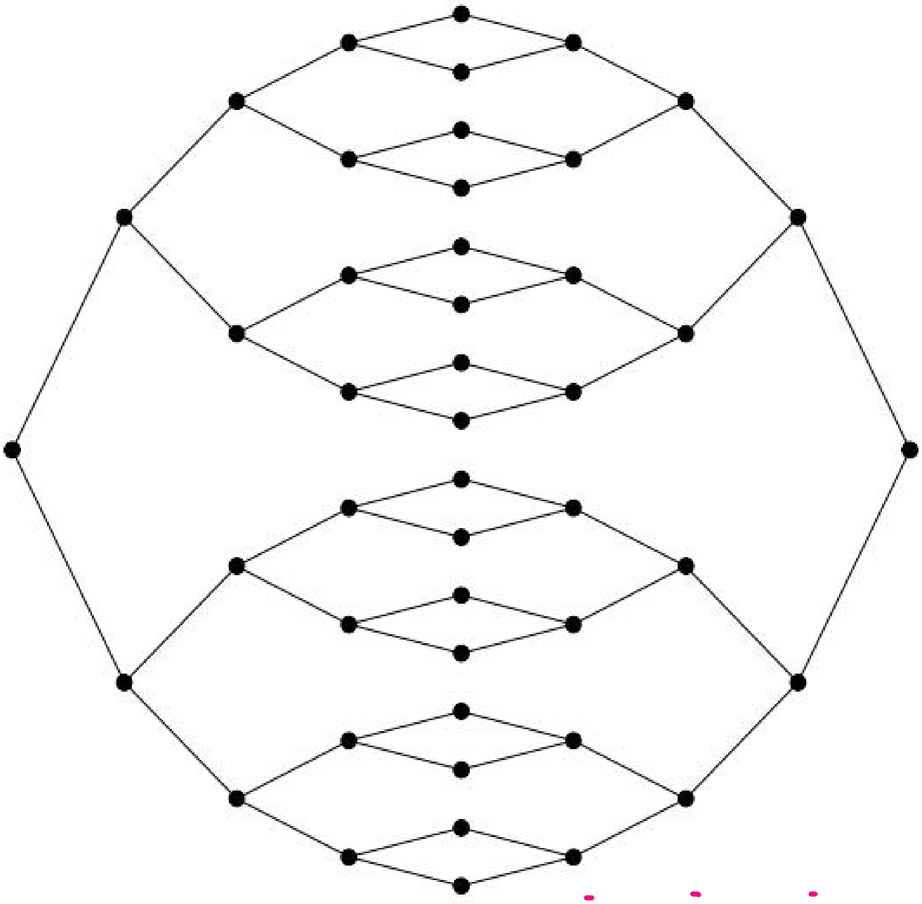
- ②
  ولگشت کلاسیک : زمان مخلوط شدن تنها به λ_1 و λ_2 بستگی دارد.
  ولگشت کوانتومی : زمان مخلوط شدن به همه λ_i ها بستگی دارد.

- سوال : آیا گراف هایی وجود دارند که برای آنها سرعت مخلوط شدن ولگشت کوانتومی به طور نمایی بیشتر از ولگشت کلاسیک باشد؟

D. Aharonov et al.

● پاسخ: نه!

Hitting Time. T



$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \bar{3} \quad \bar{2} \quad \bar{1} \quad \bar{0}$

$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8$

$2^0 \quad 2^1 \quad 2^2 \quad 2^3 \quad 2^4 \quad 2^3 \quad 2^2 \quad 2^1 \quad 2^0$

Quantum Random Walks in Computer Science.

Decision Problems : $\begin{cases} \text{Yes} \\ \text{No.} \end{cases}$

∴ بسیاری از مسائل را می‌توان به مسائل تصمیم‌گیری تقلیل داد. (در زمان چندجمله‌ای).

- مثال : TSP ← آیا مسیری وجود دارد که همه شهرها را طی کند و طول آن کمتر از ℓ باشد ؟

Search in an exponentially Large set S:

$$S = \{ (x_1 x_2 x_3 \ldots\ldots x_n) \;,\; x_i = 0,1 \} \qquad |S| = 2^n$$

- exact Cover Problem.

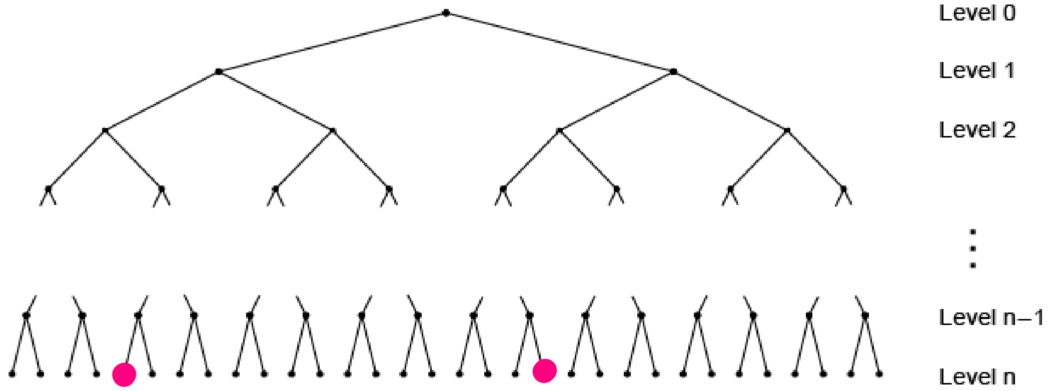


FIG. 1. The underlying branching tree. At level m there are 2^m nodes.

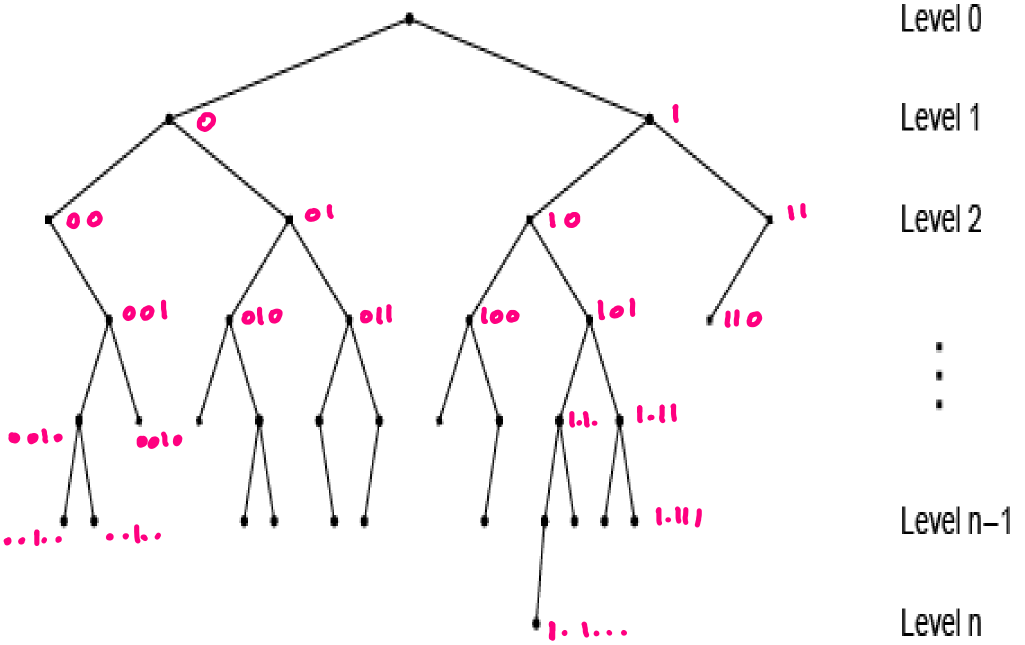
Exact Cover مثال:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}_{m \times n} \qquad \sum_{j=1}^{n} A_{ij} x_j = 1 \quad \forall i$$

if $A_{i_1} = A_{i_2} = 1 \rightarrow (1\ 1\ \ldots\ldots)$ excluded.

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}_{m \times n}$$

$$x_2 + x_4 + x_5 + x_7 = 1$$
$$x_2 + x_3 + x_5 + x_7 = 1$$
$$x_2 + x_3 + x_5 = 1$$
$$x_1 + x_3 + x_6 = 1$$



Random Walk in the space of solutions :

Other Applications of CRW
- تخمین حجم یک شکل محدب
- محاسبهٔ $|M|_+$ برای یک ماتریس M
- پیدا کردن یک جملهٔ صدق برای یک بابت

$x_1, x_2, x_3, \dots\dots\dots x_N$

$$F = C_1 \wedge C_2 \wedge \dots\dots \wedge C_M$$

Algorithm: For 2-SAT $\qquad C = x_i^{\pm} \vee x_j^{\pm}$

۱- یک رشتهٔ تصادفی $X = (0110110001101)$ انتخاب می‌کنیم و جملات C_1، C_2، .... تا C_m را یکی یکی امتحان می‌کنیم.

۲- اگر به یک جملهٔ زشت رسیدیم، مثل $C' = x_i^{\pm} \vee x_j^{\pm}$، یکی از x_i یا x_j را به طور تصادفی برمی‌گردانیم.

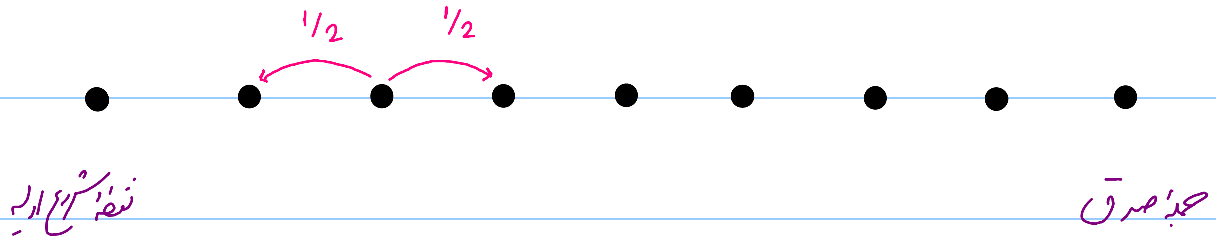
	$x_i \vee x_j$	$x_i \vee \bar{x}_j$	$\bar{x}_i \vee x_j$	$\bar{x}_i \vee \bar{x}_j$
00	*	✓	✓	✓
01	✓	*	✓	✓
10	✓	✓	*	✓
11	✓	✓	✓	*

اگر متغیر صحیح را برداریم، فاصله X با جواب صدق یکی کم می‌شود.

اگر متغیر غیرصحیح را برداریم، فاصله X با جواب صدق یکی زیاد می‌شود.

$$\begin{tikzcd} & 00 & P = 1/2 \\ 01 \arrow[ur] \arrow[dr] & & \\ & 11 & P = 1/2 \end{tikzcd}$$

بنابراین با یک ولگشت در یک ابرمکعب n بعدی سروکار داریم که احتمال گذار از هر لایه به لایه دیگر (برحسب فاصله Hamming) $\frac{1}{2}$ است.



Hitting Time $T \simeq n^2$.